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Inequalities for Trigonometric Polynomials

A. K. VARMA

Department of Mathematics, University of Florida, Gainesville, Florida 32611, U.S.A.

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Let $t_n(x)$ be any real trigonometric polynomial of degree n such that $\|t_n\|_{L_\infty} \leqslant 1$. Here we are concerned with obtaining the best possible upper estimate of

$$\int_0^{2\pi} (t_n^{(k)}(x))^{2r+2}\, dx \Big/ \int_0^{2\pi} (t_n^{(k)}(x))^{2r}\, dx$$

where $r \geqslant 0$ (integer). As a special case we obtain (1.2). Let $\|t_n\|_{L_2}$ and $\|t_n^{(r)}\|_{L_2}$ be given where t_n is any real trigonometric polynomial. In Theorem 2 we shall obtain the estimate of $\|t_n^{(j)}\|_{L_2}$ in terms of $\|t_n\|_{L_2}$ and $\|t_n^{(r)}\|_{L_2}$. The results are again the best possible.

1. INTRODUCTION

A. P. Calderon and G. Klein [1] proved the following theorem for trigonometric polynomials:

THEOREM A. *Suppose that $\varphi(x)$ is a nonnegative function defined for nonnegative x and satisfies the condition that $(\varphi(x) - \varphi(0))/x$ be a non-decreasing function of x, $x \geqslant 0$. Then the maximum of the integral $\int_0^{2\pi} \phi(|s_n'(x)|)\, dx$ for all trigonometric polynomials $s_n(x)$ of order n, bounded in absolute value by 1, is achieved by the Chebyshev polynomial $\cos(nx+\alpha)$. If in addition $\varphi(x)$ is not a constant function, then the Chebyshev polynomial is the only such polynomial achieving this maximum.*

Later Theorem A of Calderon and Klein [1] was extended by L. V. Taikov [4] and also by G. K. Kristiansen [2]. Under the same conditions imposed on $\varphi(x)$ and $s_n(x)$ as stated in Theorem A, G. K. Kristiansen proved (Theorem C of [2]) that

$$\int_0^{2\pi} \varphi(n^{-k} |s_n^{(k)}(x)|)\, dx \leqslant \int_0^{2\pi} \varphi(|\sin x|)\, dx. \tag{1.1}$$

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In particular, we have

$$\int_0^{2\pi} (s_n^{(k)}(x))^2 \, dx \leqslant \pi n^{2k} \tag{1.2}$$

with equality for $s_n(x) = \cos(nx + \alpha)$.

2. Main Results

The object of this paper is to prove the following inequality related to trigonometric polynomials and also obtain (1.2) as a special case. We now state

Theorem 1. *If $t_n(x)$ is a real trigonometric polynomial of order n such that* $\max_{0 \leqslant x \leqslant 2\pi} |t_n(x)| \leqslant 1$, *and r is any nonnegative integer, then we have*

$$\int_0^{2\pi} (t_n^{(k)}(x))^{2r+2} \, dx \leqslant \frac{(2r+1)}{(2r+2)} n^{2k} \int_0^{2\pi} (t_n^{(k)}(x))^{2r} \, dx, \tag{2.1}$$

with equality for $t_n(x) = \cos(nx + \alpha)$. Moreover, with repeated application of (2.1) we have

$$\int_0^{2\pi} (t_n^{(k)}(x))^{2r+2} \, dx \leqslant \frac{(2r+1)(2r-1)\cdots 5 \cdot 3 \cdot 1}{(2r+2)(2r)\cdots 4 \cdot 2} n^{(2r+2)k} 2\pi. \tag{2.2}$$

Remark. Putting $r = 0$ in (2.1) we obtain (1.2). Also, (2.2) is a special case of (1.1).

The object of the next theorem is to consider the following problem related to trigonometric polynomials.

We define

$$\|t_n\|_{L_2}^2 = \frac{1}{2\pi} \int_0^{2\pi} (t_n(x))^2 \, dx. \tag{2.3}$$

Next, suppose r is a fixed positive integer such that $r \geqslant 2$. Given $\|t_n\|_{L_2}$ and $\|t_n^{(r)}\|_{L_2}$, the problem is to obtain the estimate of $\|t_n^{(j)}\|_{L_2}$ for $j = 1, 2, ..., r-1$. In this direction we shall prove

Theorem 2. *Let $t_n(x)$ be a given real trigonometric polynomial of order $\leqslant n$. Then we have*

$$r \|t_n^{(j)}\|_{L_2}^2 \leqslant (r-j) n^{2j} \|t_n\|_{L_2}^2 + j n^{2j-2r} \|t_n^{(r)}\|_{L_2}^2 \qquad j = 1, 2, ..., r-1. \tag{2.4}$$

Moreover, equality holds for $t_n(x) = \cos(nx + \alpha)$.

Remark. Theorem 2 is motivated by the earlier result of the author in [5] and references mentioned in it.

3. Preliminary

For the proof of Theorem 1, we shall need the following

LEMMA 3.1. *Let $t_n(x)$ be an arbitrary real trigonometric polynomial of order n such that* $\max_{0 \leqslant x \leqslant 2\pi} |t_n(x)| \leqslant 1$. *Then we have*

$$(t_n^{(k)}(x))^2 - t_n^{(k-1)}(x)\, t_n^{(k+1)}(x) \leqslant n^{2k}, \qquad 0 \leqslant x \leqslant 2\pi. \tag{3.1}$$

Equality is attained for $t_n(x) = \cos(nx + \alpha)$.

Proof. The proof of this lemma is based on an interesting inequality (G. Szegő [3]). It states that if $f_n(x)$ is a real trigonometric polynomial of order n such that $|f_n(x)| \leqslant 1$, $0 \leqslant x \leqslant 2x$, then

$$n^2 f_n^2(x) + (f_n'(x))^2 \leqslant n^2, \tag{3.2}$$

where equality holds for $f_n(x) = \cos(nx + \alpha)$.

Since $|t_n(x) \leqslant 1$, $0 \leqslant x \leqslant 2\pi$ it follows from Bernstein's inequality that, for $0 \leqslant x \leqslant 2\pi$,

$$|t_n^{(k-1)}(x)| \leqslant n^{k-1}.$$

Now, we set $f_n(x) = t_n^{(k-1)}(x)/n^{k-1}$.
Clearly $|f_n(x)| \leqslant 1$ for $0 \leqslant x \leqslant 2\pi$.
Hence, by using (3.2) we obtain

$$n^2 \left(\frac{t_n^{(k-1)}(x)}{n^{k-1}} \right)^2 + \left(\frac{t_n^{(k)}(x)}{n^{k-1}} \right)^2 \leqslant n^2 \tag{3.3}$$

and

$$n^2 \left(\frac{t_n^{(k)}(x)}{n^k} \right)^2 + \left(\frac{t_n^{(k+1)}(x)}{n^k} \right)^2 \leqslant n^2. \tag{3.4}$$

Next, we define

$$a = \frac{t_n^{(k-1)}(x)}{n^{k-1}}, \qquad b = \frac{t_n^{(k)}(x)}{n^k}, \qquad \text{and} \qquad c = \frac{t_n^{(k+1)}(x)}{n^{k+1}}. \tag{3.5}$$

Then (3.3) and (3.4) are respectively equivalent to $a^2+b^2\leqslant 1$, $b^2+c^2\leqslant 1$.
Thus it easily follows that $b^2-ac\leqslant 1$.
Replacing the values of a, b, c from (3.5) we obtain (3.1).

4. Proof of Theorem 1

On multiplying (3.1) by $(t_n^{(k)}(x))^{2r}$ and integrating both sides from 0 to 2π we obtain

$$\int_0^{2\pi} (t_n^{(k)}(x))^{2r+2}\,dx - \int_0^{2\pi} t_n^{(k+1)}(x)\,t_n^{(k-1)}(x)(t_n^{(k)}(x))^{2r}\,dx \leqslant n^{2k}\int_0^{2\pi} (t_n^{(k)}(x))^{2r}\,dx. \tag{4.1}$$

Also, we note that

$$\begin{aligned}\int_0^{2\pi} (t_n^{(k)}(x))^{2r+2}\,dx &= \int_0^{2\pi} (t_n^{(k)}(x))^{2r+1}\,t_n^{(k)}(x)\,dx\\ &= -(2r+1)\int_0^{2\pi} (t_n^{(k)}(x))^{2r}\,t_n^{(k-1)}(x)\,t_n^{(k+1)}(x)\,dx.\end{aligned} \tag{4.2}$$

From (4.1) and (4.2) we obtain

$$\int_0^{2\pi} (t_n^{(k)}(x))^{2r+2}\,dx + \frac{1}{(2r+1)}\int_0^{2\pi} (t_n^{(k)}(x))^{2r+2}\,dx \leqslant n^{2k}\int_0^{2\pi} (t_n^{(k)}(x))^{2r}\,dx. \tag{4.3}$$

But (4.3) is clearly equivalent to (1.4). This completes the proof of Theorem 1.

5. Proof of Theorem 2

The proof of this theorem is given by induction. Let us define

$$A_j = \int_0^{2\pi} (t_n^{(j)}(x))^2\,dx, \tag{5.1}$$

where $t_n(x)$ is any trigonometric polynomial of order n. The main idea behind the proof is the identity

$$A_j = \frac{1}{2n^2} A_{j+1} + \frac{n^2}{2} A_{j-1} - \frac{1}{2n^2} \int_0^{2\pi} (t_n^{(j+1)}(x) + n^2 t_n^{(j-1)}(x))^2 \, dx \qquad j = 1, 2, ..., \tag{5.2}$$

which is easy to verify. From (5.2) we obtain

$$A_j \leqslant \frac{1}{2n^2} A_{j+1} + \frac{n^2}{2} A_{j-1} \qquad j = 1, 2... \tag{5.3}$$

with equality for $t_n(x) = \cos(nx + \alpha)$.

From (5.3) follows (2.4) for the case $r = 2$, $j = 1$. Next, we note that

$$A_1 \leqslant \frac{1}{2n^2} A_2 + \frac{n^2}{2} A_0 \leqslant \frac{1}{2n^2} \left\{ \frac{1}{2n^2} A_3 + \frac{n^2}{2} A_1 \right\} + \frac{n^2}{2} A_0.$$

On rearranging, we have

$$A_1 \leqslant \frac{A_3}{3} n^{-4} + \frac{2}{3} n^2 A_0. \tag{5.4}$$

Similarly, from (5.3) and (5.4), we have

$$A_2 \leqslant \frac{1}{2n^2} A_3 + \frac{n^2}{2} A_1 \leqslant \frac{1}{2n^2} A_3 + \frac{n^2}{2} \left(\frac{A_3}{3n^4} + \frac{2}{3} n^2 A_0 \right).$$

From this we have

$$A_2 \leqslant \frac{2}{3n^2} A_3 + \frac{n^4}{3} A_0. \tag{5.5}$$

(5.4) and (5.5) prove (2.4) for $r = 3$ and $j = 1, 2$, respectively. Next, we assume that for a given r (2.4) is valid for $j = 1, 2, ..., r - 1$. Now, by using (5.3),

$$A_r \leqslant \frac{1}{2n^2} A_{r+1} + \frac{n^2}{2} A_{r-1} \leqslant \frac{1}{2n^2} A_{r+1} + \frac{n^2}{2} \left(\frac{1}{r} n^{2r-2} A_0 + \frac{r-1}{r} n^2 A_r \right).$$

On rearrangement, we have

$$A_r \leqslant \left(\frac{r}{r+1}\right)\frac{1}{n^2}A_{r+1} + \frac{n^{2r}}{r+1}A_0.$$

Similarly, we obtain the upper estimate of $A_{r-1}, A_{r-2}, ..., A_2, A_1$ in terms of A_{r+1} and A_0. This proves Theorem 2 as well.

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